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From One-Component KP Hierarchy to Two-Component KP Hierarchy and Back

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Abstract

We show that the system of the standard one-component KP hierarchy endowed with a special infinite set of abelian additional symmetries, generated by squared eigenfunction potentials, is equivalent to the two-component KP hierarchy.

Background Information on the KP Hierarchy and Ghosts Symmetries.

The starting point of our presentation is the pseudo-differential Lax operator $\mathcal{L}$ obeying KP evolution equations w.r.t. the multi-time $(t) \equiv (t_1 \equiv x, t_2, \ldots)$:

$$\mathcal{L} = D + \sum_{i=1}^{\infty} u_i D^{-i} \quad ; \quad \frac{\partial \mathcal{L}}{\partial t_l} = \left[\left(\mathcal{L}^l\right)_+ , \mathcal{L} \right] \quad , \quad l = 1, 2, \ldots \tag{1}$$

The symbol D stands for the differential operator $\partial/\partial x$, whereas $\partial \equiv \partial_x$ will denote derivative of a function. Equivalently, one can represent Eq.(1) in terms of the dressing operator W whose pseudo-differential series are expressed in terms of the so called tau-function $\tau(t)$:

$$\mathcal{L} = WDW^{-1} \quad , \quad \frac{\partial W}{\partial t_l} = -\left(\mathcal{L}^l\right)_- W \quad , \quad W = \sum_{n=0}^{\infty} \frac{p_n\left(-[\partial]\right)\tau(t)}{\tau(t)} D^{-n} \tag{2}$$

with the notation: $[y] \equiv (y_1, y_2/2, y_3/3, \ldots)$ for any multi-variable $(y) \equiv (y_1, y_2, y_3, \ldots)$ and with $p_k(y)$ being the Schur polynomials. In the present approach a basic notion is that of (adjoint) eigenfunctions $\Phi(t)$, $\Psi(t)$ of the KP hierarchy satisfying :

$$\frac{\partial \Phi}{\partial t_k} = \mathcal{L}^k_+ (\Phi) \qquad ; \qquad \frac{\partial \Psi}{\partial t_k} = - \left(\mathcal{L}^*\right)^k_+ (\Psi) \tag{3}$$

The Baker-Akhiezer (BA) "wave" functions $\psi_{BA}(t,\lambda) = W(\exp(\xi(t,\lambda)))$ and its adjoint $\psi^*_{BA}(t,\lambda) = (W^*)^{-1}(\exp(-\xi(t,\lambda)))$ (with $\xi(t,\lambda) \equiv \sum_{l=1}^{\infty} t_l \lambda^l$) are (adjoint) eigenfunctions satisfying additionally the spectral equations $\mathcal{L}^{(*)}\left(\psi^{(*)}_{BA}(t,\lambda)\right) = \lambda \psi^{(*)}_{BA}(t,\lambda)$.

Throughout this note we will rely on an important tool provided by the spectral representation of eigenfunctions [1]. The spectral representation is equivalent to the following statement. Φ and Ψ are (adjoint) eigenfunctions if and only if they obey the integral representation:

$$\Phi(t) = \int dz \, \frac{e^{\xi(t-t',z)}}{z} \frac{\tau(t-[z^{-1}])\tau(t'+[z^{-1}])}{\tau(t)\tau(t')} \Phi\left(t'+[z^{-1}]\right) \tag{4}$$

$$\Psi(t) = \int dz \, \frac{e^{\xi(t'-t,z)}}{z} \frac{\tau(t+[z^{-1}])\tau(t'-[z^{-1}])}{\tau(t)\tau(t')} \Psi(t'-[z^{-1}]) \tag{5}$$

where $\int dz$ denotes contour integral around origin.

One needs to point out that the proper understanding of Eqs.(4) and (5) (as in the case of original Hirota bilinear identities) requires, following [2], expanding of the integrand in (4) and (5) as formal power series w.r.t. in $t'_n - t_n$, $n = 1, 2, \ldots$.

Consider now an infinite system of independent (adjoint) eigenfunctions $\{\Phi_j, \Psi_j\}_{j=1}^{\infty}$ of the standard KP hierarchy Lax operator $\mathcal{L}$ and define the following infinite set of the additional "ghost" symmetry flows [3]:

$$\frac{\partial}{\partial \bar{t}_s}\mathcal{L} = \Big[\mathcal{M}_s \, , \, \mathcal{L}\Big] \quad , \quad \mathcal{M}_s = \sum_{j=1}^{s} \Phi_{s-j+1} D^{-1} \Psi_j \tag{6}$$

$$\frac{\partial}{\partial \bar{t}_s}\Phi_k = \sum_{j=1}^{s} \Phi_{s-j+1} S_{k,j} - \Phi_{k+s} \quad ; \quad \frac{\partial}{\partial \bar{t}_s}\Psi_k = \sum_{j=1}^{s} \Psi_j S_{s-j+1,k} + \Psi_{k+s} \tag{7}$$

$$\frac{\partial}{\partial \bar{t}_s} F = \sum_{j=1}^{s} \Phi_{s-j+1} S\left(F \Psi_j\right) \quad ; \quad \frac{\partial}{\partial \bar{t}_s} F^* = \sum_{j=1}^{s} \Psi_j S\left(\Phi_{s-j+1} F^*\right) \tag{8}$$

where $s, k = 1, 2, \ldots$, and F (F^*) denote generic (adjoint) eigenfunctions which do not belong to the "ghost" symmetry generating set $\{\Phi_j, \Psi_j\}_{j=1}^{\infty}$. Moreover, we used abbrevations $S_{k,j} \equiv S\left(\Phi_k, \Psi_j\right) = \partial^{-1}\left(\Phi_k \Psi_j\right)$ to denote the so called squared eigenfunction potentials (SEP) [4, 1] for which we find the "ghost" symmetry flows :

$$\frac{\partial}{\partial \bar{t}_s} S_{k,l} = S_{k,l+s} - S_{k+s,l} + \sum_{j=1}^{s} S_{k,j} S_{s-j+1,l} \tag{9}$$

Eqs.(7) become for the first "ghost" symmetry flow $\bar{\partial} \equiv \partial / \partial \bar{t}_1$:

$$\bar{\partial}\Phi_k = \Phi_1 S_{k,1} - P_{k+1} \quad , \quad \bar{\partial}\Psi_k = \Psi_1 S_{1,k} + \Psi_{k+1} \quad ; \quad \bar{\partial} F = \Phi_1 \partial^{-1}\left(\Psi_1 F\right) \tag{10}$$

It is easy to show that the "ghost" symmetry flows $\partial / \partial \bar{t}_s$ from Eqs.(6)-(8) commute. This can be done by proving that the ∂-pseudo-differential operators $\mathcal{M}_s$ (6) satisfy the zero-curvature equations $\partial \mathcal{M}_r / \partial \bar{t}_s - \partial \mathcal{M}_s / \partial \bar{t}_r - \left[\mathcal{M}_s \, , \, \mathcal{M}_r\right] = 0$.

Lax Representation for the Ghosts Flows

We now show that the "ghost" symmetry flows from Eqs. (6)-(8) admit their own Lax representation in terms of the pseudo-differential Lax operator $\bar{\mathcal{L}}$ w.r.t. multi-time $(\bar{t}) \equiv (\bar{t}_1 \equiv \bar{x}, \bar{t}_2, \ldots)$. While showing it we will make contact with the structure defining the affine coordinates on the Universal Grassmannian Manifold (UGM) [5, 6]. For this purpose we define objects:

$$\bar{w}_i = \frac{\Phi_{i+1}}{\Phi_1} \quad ; \quad i = 0, 1, 2, \ldots \tag{11}$$

which can be grouped together into the Laurent series expansion:

$$\bar{w} = \sum_{i=0}^{\infty} \frac{\Phi_{i+1}}{\Phi_1} z^{-i} = 1 + \sum_{l=1}^{\infty} \frac{\Phi_{l+1}}{\Phi_1} z^{-l} \tag{12}$$

From Eq.(7) we find that the action of the "ghost" symmetry flows on $\bar{w}_i$ takes a form:

$$\frac{\partial}{\partial \bar{t}_s} \bar{w}_k = -\bar{w}_{k+s} + \sum_{l=0}^{s} \bar{w}_l \overline{W}_k^{(s-l)} \tag{13}$$

where the coefficients $\overline{W}_k^{(j)}$ are given in terms of SEP-functions as

$$\overline{W}_k^{(j)} = S_{k+1,j} - \bar{w}_k S_{1,j} \quad ; \quad j = 1, 2, \ldots, k = 0, 1, 2, \ldots \tag{14}$$

$$\overline{W}_k^{(0)} = \bar{w}_k = \frac{\Phi_{k+1}}{\Phi_1} \; . \tag{15}$$

They in turn satisfy the following flow equations resulting from those in (9) :

$$\frac{\partial}{\partial \bar{t}_s} \overline{W}_k^{(j)} = \overline{W}_k^{(j+s)} - \overline{W}_{k+s}^{(j)} + \sum_{l=1}^{s} \overline{W}_k^{(s-l)} \overline{W}_l^{(j)} \tag{16}$$

which provide an example of the matrix Riccati equations (see *e.g.* [6]). The coefficients $\overline{W}_k^{(j)}$ span the Laurent series :

$$\overline{W}^{(j)} = z^j + \sum_{l=1}^{\infty} \overline{W}_l^{(j)} z^{-l} \quad ; \quad j = 0, 1, 2, \ldots \tag{17}$$

whose structure is reminiscent of the Laurent series defining the Sato Grassmannian. The connection to the usual KP setup can now be established as follows. We first introduce the well-known notion of a long derivative :

$$\overline{\nabla}_s = \frac{\partial}{\partial \bar{t}_s} + z^s = e^{-\xi(\bar{t},z)} \frac{\partial}{\partial \bar{t}_s} e^{\xi(\bar{t},z)} \tag{18}$$

which together with Eq.(17) allow us to cast both Eqs.(16) and (13) in a more compact form:

$$\overline{\nabla}_s \bar{w} = \sum_{l=0}^{s} \bar{w}_l \overline{W}^{(s-l)} \tag{19}$$

$$\overline{\nabla}_s \overline{W}^{(j)} = \overline{W}^{(j+s)} + \sum_{l=1}^{s} \overline{W}_l^{(j)} \overline{W}^{(s-l)} \quad ; \quad j = 1, 2, \ldots \tag{20}$$

From Eq.(20) we obtain reccursive expressions for $\overline{W}^{(j)}$ with $j > 0$ in terms of non-negative powers of $\overline{\nabla}_1$ acting on $\bar{w}$. Indeed, from (20) with $j = 0$ one finds $\overline{W}^{(1)} = \overline{\nabla}_1 \bar{w} - \bar{w}_1 \bar{w}$ and so on. Finally, by increasing j one arrives at expansion $\overline{W}^{(j)} = \sum_{l=0}^{j} v_l^{(j)} \overline{\nabla}_1^l \bar{w}$, which allows to rewrite Eq.(19) as $\overline{\nabla}_s \bar{w} = \sum_{l=0}^{s} U_l^{(s)} \overline{\nabla}_1^l \bar{w}$ with some coefficients $U_l^{(s)}$. Using Eq.(18) we obtain the standard evolution equation $\partial \overline{\psi}_{BA}(\bar{t}, z)/\partial \bar{t}_s = \overline{B}_s \overline{\psi}_{BA}(\bar{t}, z)$ with $\overline{B}_s = \sum_{l=0}^{s} U_l^{(s)} \bar{D}^l$ and the wave-function :

$$\overline{\psi}_{BA}(\bar{t}, z) = \bar{w} e^{\xi(\bar{t}, z)} = \overline{\mathcal{W}} e^{\xi(\bar{t}, z)} \;\; ; \;\; \overline{\mathcal{W}} \equiv 1 + \sum_{l=1}^{\infty} \frac{\Phi_{l+1}}{\Phi_1} \bar{D}^{-l} \tag{21}$$

with $\bar{D} \equiv \partial/\partial \bar{t}_1$. The evolution operators $\overline{B}_s$ satisfy $\partial \overline{\mathcal{W}}/\partial \bar{t}_s = \overline{B}_s \overline{\mathcal{W}} - \overline{\mathcal{W}} \bar{D}^s$ and are, therefore, reproduced by the usual relation $\overline{B}_s = \left(\overline{\mathcal{W}} \bar{D}^s \overline{\mathcal{W}}^{-1} \right)_+$.

The standard KP Lax operator construction follows now upon defining the $\bar{\partial}$-Lax operator $\bar{\mathcal{L}} \equiv \overline{\mathcal{W}} \bar{D} \overline{\mathcal{W}}^{-1} = \bar{D} + \sum_{i=1}^{\infty} \bar{u}_i \bar{D}^{-i}$, which enters the hierarchy equations $\partial \bar{\mathcal{L}}/\partial \bar{t}_s = \left[\bar{\mathcal{L}}_+^s \, , \, \bar{\mathcal{L}} \right]$ with $\overline{B}_s = \bar{\mathcal{L}}_+^s$. In this way we arrive at a new integrable system defined in terms of two Lax operators $\mathcal{L}$ and $\bar{\mathcal{L}}$ with two different sets of evolution parameters t and $\bar{t}$ which we will call *double KP* system. The double KP system can be viewed as ordinary one-component KP hierarchy Eq.(2) supplemented by infinite-dimensional additional symmetry structure given by Eqs.(6)-(8).

Let $\bar{\tau}(t, \bar{t})$ be a tau-function associated with the $\bar{\partial}$-Lax operator $\bar{\mathcal{L}}$, then the following results follow from the above discussion:

$$\bar{\tau}(t, \bar{t}) = \Phi_1(t, \bar{t}) \, \tau(t, \bar{t}) \qquad , \qquad \frac{p_s \left(-[\bar{\partial}] \right) \bar{\tau}}{\bar{\tau}} = \frac{\Phi_{s+1}}{\Phi_1} \;\; ; \;\; s = 0, 1, 2, \ldots \tag{22}$$

where the $\tau(t, \bar{t})$ is tau-function of the original ∂-Lax operator $\mathcal{L}$. Moreover, for any generic eigenfunction F of $\mathcal{L}$, which does not belong to the set $\{\Phi_j\}$ in (6) and has "ghost" symmetry flows given by Eq.(8), the function $\bar{F} \equiv F/\Phi_1$ is automatically an eigenfunction of the "ghost" Lax operator $\bar{\mathcal{L}}$:

$$\frac{\partial}{\partial \bar{t}_s} \left(F/\Phi_1 \right) = \bar{\mathcal{L}}_+^s \left(F/\Phi_1 \right) \tag{23}$$

We will also introduce the Darboux-Bäcklund (DB) transformations:

$$\bar{\mathcal{L}}(n+1) = \left(\frac{1}{\Phi_1^{(n+1)}} \bar{D}^{-1} \Phi_1^{(n+1)} \right) \bar{\mathcal{L}}(n) \left(\frac{1}{\Phi_1^{(n+1)}} \bar{D} \Phi_1^{(n+1)} \right) \tag{24}$$

for the “ghost” KP Lax operator which have an additional property of commuting with the “ghost” symmetries (6). In the Eq.(24) the the DB “site” index (n) parametrizes the DB orbit. The convention we adopt is that the index (n) labels the particular $\bar{\partial}$-Lax operator $\bar{\mathcal{L}}$ constructed above. In terms of the original isospectral flows the DB transformations take a form :

$$\mathcal{L}(n+1) = \left(\Phi_1^{(n)} D \Phi_1^{(n)\,-1}\right) \mathcal{L}(n) \left(\Phi_1^{(n)} D^{-1} \Phi_1^{(n)\,-1}\right) \tag{25}$$

where $\mathcal{L}(n)$ is the original Lax operator underlying our construction. In this setting the tau-function $\bar{\tau}$ appears, according to Eq.(22), to be nothing but the tau-function associated with the the Lax operator $\mathcal{L}(n+1)$ at the site $(n+1)$, namely $\bar{\tau} = \tau(n+1)$.

We can now present results for the adjoint eigenfunctions Ψ_i which parallel those in Eqs.(22)-(23) for the eigenfunctions Φ_i. Defining $\bar{w}_k^* = \Psi_{k+1}/\Psi_1$ we find that :

$$\frac{\partial}{\partial \bar{t}_s} \bar{w}_k^* = \bar{w}_{k+s}^* + \sum_{l=0}^{s} \bar{w}_l^* \overline{W}^{*\,(s-l)}{}_k \tag{26}$$

with the coefficients

$$\overline{W}^{*\,(j)}{}_k = S_{j,k+1} - \bar{w}_k^* S_{j,1} \quad ; \quad j = 1,2,\ldots, k = 0,1,2,\ldots \tag{27}$$

$$\overline{W}^{*\,(0)}{}_k = \bar{w}_k^* = \frac{\Psi_{k+1}}{\Psi_1} \,. \tag{28}$$

Let $\widehat{\tau}(t,\bar{t})$ be a tau-function associated with the $\bar{\partial}$-Lax operator $\bar{\mathcal{L}}(n-2)$ at the DB site $(n-2)$. Then the following results can be shown:

$$\widehat{\tau}(t,\bar{t}) = \Psi_1(t,\bar{t})\, \tau(t,\bar{t}) \qquad , \qquad \frac{p_s\left([\bar{\partial}]\right)\widehat{\tau}}{\widehat{\tau}} = \frac{\Psi_{s+1}}{\Psi_1} \;\; ; \;\; s = 0,1,2,\ldots \tag{29}$$

where $\tau(t,\bar{t})$ is the tau-function of the original ∂-Lax operator $\mathcal{L}$ (at the DB site (n)) . Moreover, for any generic adjoint eigenfunction F^* of $\mathcal{L}$, which does not belong to the set $\{\Psi_j\}$ in (6) and satisfies, therefore, the “ghost” symmetry flows given by Eq.(8)) the function F^*/Ψ_1 is an adjoint eigenfunction of the “ghost” Lax operator $\bar{\mathcal{L}}(n-2)$:

$$\frac{\partial}{\partial \bar{t}_s} \left(F^*/\Psi_1\right) = -\left(\bar{\mathcal{L}}^*(n-2)\right)_+^s \left(F^*/\Psi_1\right) \tag{30}$$

Let us list two other important identities which relate the tau-function τ to the SEP-functions (using notation of Eqs.(6)-(8)) :

$$\frac{p_j\left([\bar{\partial}]\right)\tau}{\tau} = -S_{1,j} \quad ; \quad \frac{p_j\left(-[\bar{\partial}]\right)\tau}{\tau} = S_{j,1} \quad ; \quad j \geq 1 \tag{31}$$

Embedding of Double KP System into Two-Component KP Hierarchy

The two-component KP hierarchy [7] is given by three tau-functions $\tau_{11}, \tau_{12}, \tau_{21}$ depending on two sets of multi-time variables $t, \bar{t}$ and obeying the following Hirota bilinear identities:

$$\int dz \frac{e^{\xi(\bar{t}-\bar{t}',z)}}{z^2} \tau_{12}(t, \bar{t} - [z^{-1}])\tau_{21}(t', \bar{t}' + [z^{-1}]) =$$

$$= \int dz e^{\xi(t-t',z)} \tau_{11}(t - [z^{-1}], \bar{t})\tau_{11}(t' + [z^{-1}], \bar{t}') \tag{32}$$

$$\int dz \frac{e^{\xi(\bar{t}-\bar{t}',z)}}{z} \tau_{12}(t, \bar{t} - [z^{-1}])\tau_{11}(t', \bar{t}' + [z^{-1}]) =$$

$$= \int dz \frac{e^{\xi(t-t',z)}}{z} \tau_{11}(t - [z^{-1}], \bar{t})\tau_{12}(t' + [z^{-1}], \bar{t}') \tag{33}$$

$$\int dz \frac{e^{\xi(\bar{t}-\bar{t}',z)}}{z} \tau_{11}(t, \bar{t} - [z^{-1}])\tau_{21}(t', \bar{t}' + [z^{-1}]) =$$

$$= \int dz \frac{e^{\xi(t-t',z)}}{z} \tau_{21}(t - [z^{-1}], \bar{t})\tau_{11}(t' + [z^{-1}], \bar{t}') \tag{34}$$

$$\int dz e^{\xi(\bar{t}-\bar{t}',z)} \tau_{11}(t, \bar{t} - [z^{-1}])\tau_{11}(t', \bar{t}' + [z^{-1}]) =$$

$$= \int dz \frac{e^{\xi(t-t',z)}}{z^2} \tau_{21}(t - [z^{-1}], \bar{t})\tau_{12}(t' + [z^{-1}], \bar{t}') \tag{35}$$

We will now show that the double KP system defined in the previous section in terms of the tau-functions $\tau, \bar{\tau}, \widehat{\tau}$, will satisfy the Hirota identities (32)-(35) upon the identification :

$$\tau = \tau_{11} \quad ; \quad \bar{\tau} = \tau_{12} \quad ; \quad \widehat{\tau} = \tau_{21} \tag{36}$$

and upon making the obvious identication for the multi-time variables t and $\bar{t}$.

As an example of our method we will derive Eq.(33) using the technique which employs the spectral representations (4)-(5). Let F be a wave-function for the $\mathcal{L}$ Lax operator:

$$F = \psi_{BA}(t, \bar{t}, \lambda) = \frac{\tau(t - [\lambda^{-1}], \bar{t})}{\tau(t, \bar{t})} e^{\xi(t,\lambda)} \tag{37}$$

According to Eq.(23) :

$$\frac{F}{\Phi_1} = \frac{\tau(t - [\lambda^{-1}], \bar{t})}{\bar{\tau}(t, \bar{t})} e^{\xi(t,\lambda)} \tag{38}$$

is an eigenfunction for the Lax operator $\bar{\mathcal{L}}$ w.r.t. the multi-time $\bar{t}$ and in view of Eq.(4) admits the spectral representation :

$$\frac{F}{\Phi_1}(t, \bar{t}) = \int dz \frac{e^{\xi(\bar{t}-\bar{t}',z)}}{z} \frac{\bar{\tau}(t, \bar{t} - [z^{-1}])\tau(t, \bar{t}' + [z^{-1}])}{\bar{\tau}(t, \bar{t})\bar{\tau}(t, \bar{t}')} F(t, \bar{t}' + [z^{-1}]) \tag{39}$$

Substituting Eq.(37) into the r.h.s. of Eq.(39) and Eq.(38) into the l.h.s. of Eq.(39) we obtain:

$$\bar{\tau}(t, \bar{t}')\, \tau(t - [\lambda^{-1}], \bar{t}) = \int dz \frac{e^{\xi(\bar{t}-\bar{t}',z)}}{z}\, \bar{\tau}(t, \bar{t} - [z^{-1}])\, \tau(t - [\lambda^{-1}], \bar{t}' + [z^{-1}]) \tag{40}$$

Choose now $\bar{F}$ to be a wave-function for the Lax operator $\bar{\mathcal{L}}$:

$$\bar{F} = \overline{\psi}_{BA}(t, \bar{t}, \lambda) = \frac{\bar{\tau}(t, \bar{t} - [\lambda^{-1}])}{\bar{\tau}(t, \bar{t})} e^{\xi(\bar{t}, \lambda)} \tag{41}$$

then the function :

$$\bar{F}\,\Phi_1 = \frac{\bar{\tau}(t, \bar{t} - [\lambda^{-1}])}{\tau(t, \bar{t})}\; e^{\xi(\bar{t}, \lambda)} \tag{42}$$

is an eigenfunction for the Lax operator $\mathcal{L}$ w.r.t. the multi-time t and in view of Eq.(4) admits the spectral representation :

$$\bar{F}\,\Phi_1(t, \bar{t}) = \int dz \frac{e^{\xi(t-t', z)}}{z}\; \frac{\tau(t - [z^{-1}], \bar{t})\bar{\tau}(t' + [z^{-1}], \bar{t})}{\tau(t, \bar{t})\tau(t', \bar{t})}\; F(t' + [z^{-1}], \bar{t}) \tag{43}$$

Substituting Eq.(41) into the r.h.s. of Eq.(43) and Eq.(42) into the l.h.s. of Eq.(39) we obtain:

$$\bar{\tau}(t, \bar{t} - [\lambda^{-1}])\, \tau(t', \bar{t}) = \int dz \frac{e^{\xi(t-t', z)}}{z}\, \tau(t - [z^{-1}], \bar{t})\, \bar{\tau}(t' + [z^{-1}], \bar{t} - [\lambda^{-1}]) \tag{44}$$

Subtracting Eq.(44) from Eq.(40) indeed reproduces Hirota identity (33) with identifications $t' = t - [\lambda^{-1}]$ and $\bar{t}' = \bar{t} - [\lambda^{-1}]$. Proofs of the remaining Hirota identities (32), (34) and (35) follow along similar lines.

From Two-Component KP Hierarchy to Double KP Hierarchy

In this section we are going to show that the two-component KP hierarchy, with the two sets of multi-times $t, \bar{t}$, can be regarded as ordinary one-component KP hierarchy w.r.t. to one of the multi-times, *e.g.* t, supplemented by an infinite-dimensional abelian algebra of additional ("ghosts") symmetries, such that the second multi-time $\bar{t}$ plays the role of "ghost" symmetry flow parameters.

We first observe, by putting $\bar{t} = \bar{t}'$, $t = t'$ in Hirota identities (32) and (35), that the tau-function $\tau_{11}(t, \bar{t})$ defines two one-component KP hierarchies KP_{11} and $\overline{\mathrm{KP}}_{11}$ w.r.t. the multi-time variables t and $\bar{t}$, respectively. This is because in the limits $\bar{t} = \bar{t}'$, $t = t'$ Eqs.(32), (35) reduce to the ordinary one-component KP Hirota identities.

Let us define :

$$\Phi_1(t, \bar{t}) \equiv \frac{\tau_{12}(t, \bar{t})}{\tau_{11}(t, \bar{t})} \quad ; \quad \Psi_1(t, \bar{t}) \equiv \frac{\tau_{21}(t, \bar{t})}{\tau_{11}(t, \bar{t})} \tag{45}$$

These functions have the following important properties. Φ_1 turns out to be simultaneously an eigenfunction of the KP_{11} hierarchy and an adjoint eigenfunction of the $\overline{\mathrm{KP}}_{11}$ hierarchy. Similarly, Ψ_1 is simultaneously an adjoint eigenfunction of the KP_{11} hierarchy and an eigenfunction of the $\overline{\mathrm{KP}}_{11}$ hierarchy. The proof proceeds by showing that Φ_1 and Ψ_1 satisfy the corresponding spectral representations (4) and (5) as a result of taking special limits in Hirota identities (33) and (34).

As a consequence of these last properties we conclude that $\tau_{12} = \Phi_1 \tau_{11}$ and $\tau_{21} = \Psi_1 \tau_{11}$ are also tau-functions of KP_{11} and $\overline{\mathrm{KP}}_{11}$ hierarchies since they can be regarded as DB transformations of τ_{11}.

Next, define :

$$\Phi_j(t,\bar{t}) \equiv \frac{p_{j-1}\left(-[\bar{\partial}]\right)\tau_{12}(t,\bar{t})}{\tau_{11}(t,\bar{t})} \quad ; \quad \Psi_j(t,\bar{t}) \equiv \frac{p_{j-1}\left([\bar{\partial}]\right)\tau_{21}(t,\bar{t})}{\tau_{11}(t,\bar{t})} \quad ; \quad j \geq 1 \tag{46}$$

It turns out that Φ_j, Ψ_j are (adjoint) eigenfunctions of KP_{11}. We present the proof for Φ_j which goes as follows. Substitute $\bar{t}' = \bar{t} - [\lambda^{-1}]$ in (33). Using identity $\int dz F(z)/z(1 - z/\lambda) = F_{(-)}(\lambda)$ (where the subscript $(-)$ indicates taking the non-positive-power part of the corresponding Laurent series), the r.h.s. of (33) becomes:

$$\int dz \frac{1}{z}\frac{1}{\lambda - z}\tau_{12}(t, \bar{t} - [z^{-1}])\tau_{11}(t', \bar{t} - [\lambda^{-1}] + [z^{-1}]) = \tau_{12}(t, \bar{t} - [\lambda^{-1}])\tau_{11}(t', \bar{t}) \tag{47}$$

and the Hirota Eq.(33) simplifies now to:

$$\tau_{12}(t, \bar{t} - [\lambda^{-1}])\tau_{11}(t', \bar{t}) = \int dz \frac{e^{\xi(t-t',z)}}{z}\tau_{11}(t - [z^{-1}], \bar{t})\tau_{12}(t' + [z^{-1}], \bar{t} - [\lambda^{-1}]) \tag{48}$$

Upon expanding Eq.(48) in λ and keeping the term of the order λ^j we obtain Eq.(4) for $\Phi \to \Phi_{j+1}$, with Φ_{j+1} as defined in (46). Consequently, the functions Φ_{j+1} from Eq.(46) are eigenfunctions of the KP_{11} hierarchy. The proof for Ψ_j goes along the same lines.

Define now functions $S_{1,j}$ as $S_{1,j} \equiv p_j\left([\bar{\partial}]\right)\tau_{11}/\tau_{11}$ for $j \geq 1$ (cf. Eq.(31)). We can also define the functions $S_{k,j}$ for $k > 1$ via:

$$S_{k+1,j} = \overline{W}_k^{(j)} + \bar{w}_k \frac{p_j\left([\bar{\partial}]\right)\tau_{11}}{\tau_{11}} \tag{49}$$

where $\overline{W}_k^{(j)}$ and $\bar{w}_j$ are the known affine coordinates of UGM for $\overline{\mathrm{KP}}_{11}$ hierarchy satisfying Eqs.(13) and (16). Recall that the latter flow equations can be regarded as an equivalent definition of $\overline{\mathrm{KP}}_{11}$ hierarchy.

Now, one can show that upon substitution of definitions (49) and $\bar{w}_j = \Phi_{j+1}/\Phi_1$ (following from Eq.(46)) into the $\overline{\mathrm{KP}}_{11}$ structure Eqs.(13) and (16), the latter two equations go over into Eqs.(7) and (9) for Φ_j and $S_{k,j}$, which define an infinite-dimensional additional "ghost" symmetry structure of KP_{11} hierarchy.

This last observation shows that the isospectral evolution parameters $\bar{t}$ of the one-component $\overline{\mathrm{KP}}_{11}$ hierarchy indeed play the role of parameters of an infinite-dimensional abelian algebra of additional "ghosts" symmetries of the one-component KP_{11} hierarchy.

Conclusions and Outlook.

In the present paper we have shown that, given an ordinary one-component KP hierarchy, we can always construct a two-component KP hierarchy, embedding the original one, in the following way. We choose an infinite set of (adjoint-)eigenfunctions of the one-component KP hierarchy (such a choice is always possible due to our spectral representation theorem [1]), which we use to construct an infinite-dimensional abelian algebra of additional symmetries. The one-component KP hierarchy equipped with such additional symmetry structure turns out to be equivalent to the standard two-component KP hierarchy.

It is an interesting question for further study whether the origin of higher multi-component KP hierarchies can be similarly traced back to one-component KP hierarchy endowed with an appropriate infinite-dimensional abelian additional symmetry structure, generalizing the above construction for two-component KP hierarchy.

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